Student Number___________________

ABBOTSLEIGH

AUGUST 2008
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided with this paper.
- All necessary working should be shown in every question.

Total marks – 84

- Attempt Questions 1-7.
- All questions are of equal value.
- Answer each question in a separate writing booklet.

Outcomes assessed

Preliminary course

- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives which require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving binomials, projectiles or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Harder applications of the Mathematics course are included in this course. Thus the Outcomes from the Mathematics course are included.

Outcomes from the Mathematics course

Preliminary course

- **P2** provides reasoning to support conclusions that are appropriate to the context
- **P3** performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities
- **P4** chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques
- **P5** understands the concept of a function and the relationship between a function and its graph
- **P6** relates the derivative of a function to the slope of its graph
- **P7** determines the derivative of a function through routine application of the rules of differentiation
- **P8** understands and uses the language and notation of calculus

HSC course

- **H2** constructs arguments to prove and justify results
- **H3** manipulates algebraic expressions involving logarithmic and exponential functions
- **H4** expresses practical problems in mathematical terms based on simple given models
- **H5** applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
- **H6** uses the derivative to determine the features of the graph of a function
- **H7** uses the features of a graph to deduce information about the derivative
- **H8** uses techniques of integration to calculate areas and volumes
- **H9** communicates using mathematical language, notation, diagrams and graphs

QUESTION ONE (12 Marks) **Marks**

a) Solve for x: $3^{x+1} = 2$, expressing your answer correct to two decimal places. **2**

b) State the domain and range of the function $g(x) = \frac{1}{2}\cos^{-1}\frac{x}{2}$ **2**

c) Using the remainder theorem, or otherwise, fully factorise $6x^3 + 17x^2 - 4x - 3$ **3**

d) Use the substitution $u = 2 - x^2$ to find $\int \frac{x}{\left(2-x^2\right)^3}\,dx$ **2**

e) Solve the inequality: $\frac{2x-5}{x-4} \geq x$ **3**

QUESTION TWO (12 Marks) START A NEW BOOKLET **Marks**

a) Find $\frac{d}{dx}\left(3x^2\cos^{-1}x\right)$ **2**

b) Evaluate exactly: $\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$ **2**

c) Find, in degrees and minutes, the acute angle between the lines **2**
$y = 2x + 3$ and $x - y = 1$

d) Given $(1 + 2x)^6(1 - x)^4$, find the coefficient of x^3 **2**

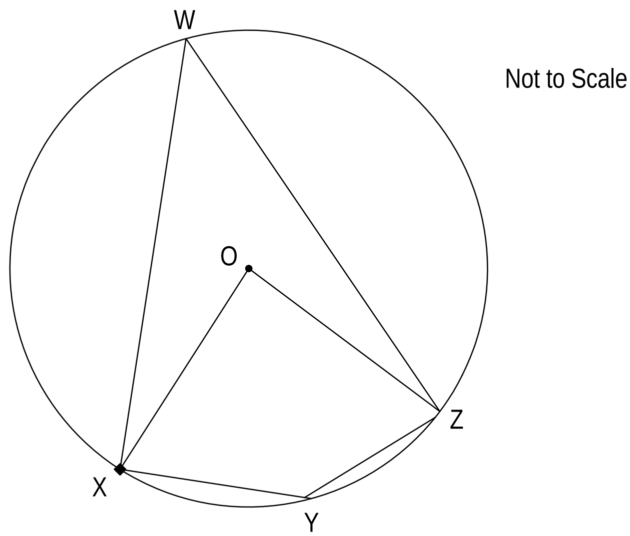
e) Use the method of mathematical induction to prove that **4**

$$2^{2n} + 8 \text{ is divisible by } 6, \quad n \geq 1$$

QUESTION THREE (12 Marks) START A NEW BOOKLET **Marks**

a) Use the Table of Standard Integrals to show that $\int_6^{15} \frac{dx}{\sqrt{x^2+64}} = \log_e 2$ **2**

b)



WXYZ is a quadrilateral inscribed in a circle with centre O. $\angle XWZ = 32^o$.
Find, giving reasons, the size of:

i) $\angle XOZ$ **1**

ii) $\angle XYZ$ **2**

c) Find the roots of $4x^3 - 4x^2 - 29x + 15 = 0$, given that the difference between two of the roots is the value of the third root. **3**

d) The population of Nottingtown first reached 25 000 on January 1st 2000. Nottingtown's population is predicted to increase according to the equation

$$\frac{dN}{dt} = k\left(N - 8000\right)$$

Where t represents the time in years after the population first reached 25 000.

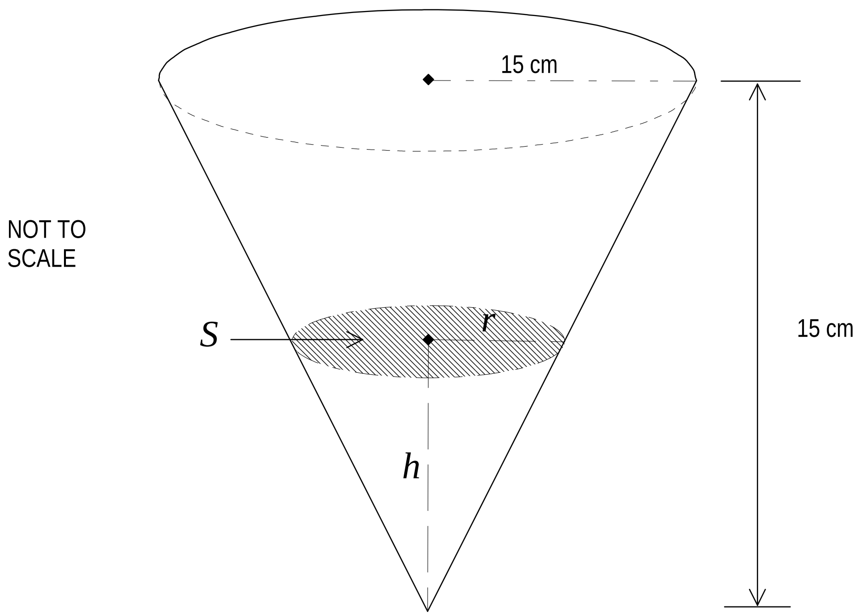
On January 1st 2005, the population of Nottingtown was 29 250.

i) Show that $N = 8000 + Ae^{kt}$ where A is a constant, is a solution to the above equation **2**

ii) Calculate the values of A and k. **2**

QUESTION FOUR (12 Marks) START A NEW BOOKLET **Marks**

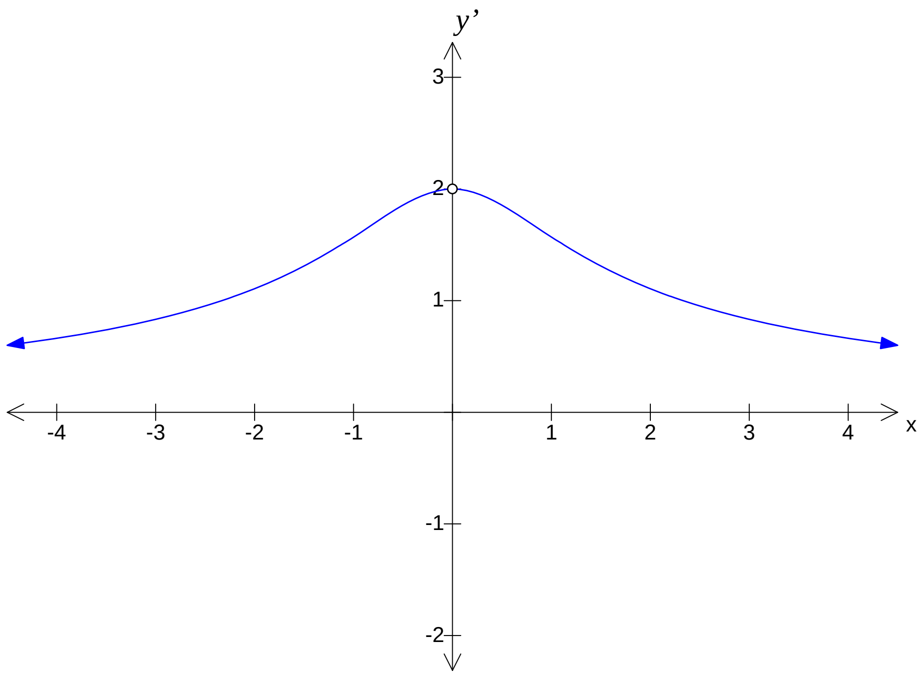
a) Water is poured into a cone of radius 15 cm and height 15 cm. The water is poured in at a constant rate of 12 cm^3/s. The depth of the water at t seconds is h cm.



i) The shaded area, S, represents the surface of the water as the cone is filled. Given r is the radius of area S, show that the radius is increasing at $\frac{12}{\pi r^2}$ cms^{-1}. **3**

ii) Hence, calculate the rate at which the surface area, S, of the water is changing when the depth of the water is 5 cm. **3**

b) The graph below shows the derivative of $y = 2\tan^{-1}x$



i) Where does $y = 2\tan^{-1}x$ have the greatest slope and what is its value? **2**

ii) What x values correspond to $y' = \dfrac{1}{3}$ **2**

iii) Deduce the limiting sum bounded by the gradient function, the x-axis and $-\infty < x < \infty$ **2**

QUESTION FIVE (12 Marks) START A NEW BOOKLET **Marks**

a) Differentiate $y = 2^x$ with respect to x. **1**

b) A particle moves in a straight line with an acceleration given by

$$\frac{d^2x}{dt^2} = 9(x-2)$$

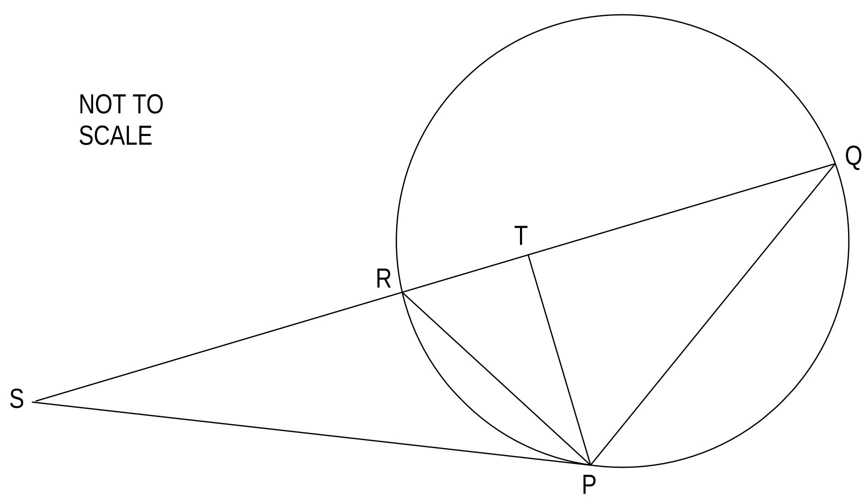
where x is the displacement in metres from the origin, O, after t seconds. Initially the particle is 4 metres to the right of O and has velocity, $v = -6$.

i) Show that $v^2 = 9(x-2)^2$ **2**

ii) Find an expression for v and hence find x as a function of t **3**

iii) Explain whether the velocity of the particle is ever zero **2**

c)



QR is the diameter of the circle. The tangent to the circle at P meets QR produced to S. T is situated on QR such that PR bisects ∠TPS.

COPY OR TRACE THE DIAGRAM ONTO YOUR PAGE

i) Give a reason why ∠RPS = ∠PQR **1**

ii) Hence, show that PT ⊥ QR **3**

QUESTION SIX (12 Marks) START A NEW BOOKLET **Marks**

a) Consider the function $y = x(x-2)^2$, $x \le a$ where a is a constant.

i) Find the values of a , given that the inverse function, $y = f^{-1}(x)$ exists. **2**

ii) State the domain of $y = f^{-1}(x)$ **1**

b) Show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 3x \, dx = \frac{1}{2}\left(\frac{\pi}{12} - \frac{1}{6}\right)$ **3**

c) A golfer hits a ball so that it clears a tree which is 6 metres in height and with a horizontal distance of 20 metres (assuming the ground is level). If the selected club produces an angle of elevation of 40^o (given g = 10 ms^{-2}),

i) Write an expression for y, the vertical distance travelled. **1**

ii) Write an expression for x, the horizontal distance travelled. **1**

iii) Hence, determine the equation of the flight path (in terms of x and y). **2**

iv) Calculate the speed at which the golf ball must leave the ground to ensure it just clears the tree. **2**

QUESTION SEVEN (12 Marks) START A NEW BOOKLET **Marks**

a) Find: $\int_0^{1.25} \frac{5\,dx}{\sqrt{25-16x^2}}$ **2**

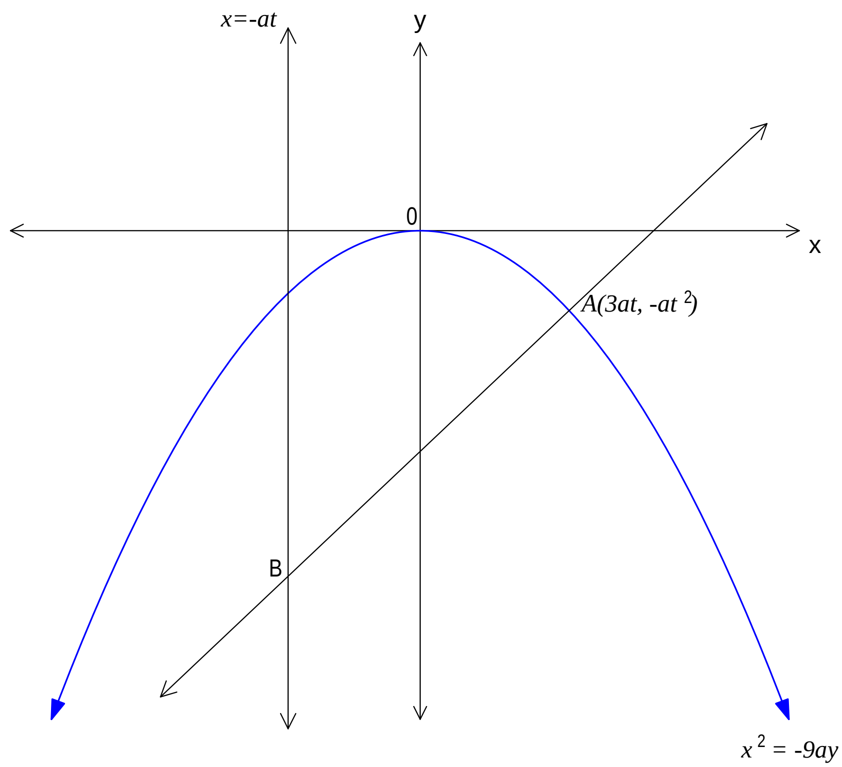
b) Given that a root for the equation $e^x - x - 2 = 0$ is close to $x = 1.2$, use one application of Newton's Method to find a second approximation for this root, correct to 2 decimal places. **2**

Question Seven continued over page…

c) The point A($3at$, $-at^2$) is a variable point on the parabola $x^2 = -9ay$.

The normal at A meets the line $x = -at$ at point B.

Point C lies on the normal and divides interval AB externally in the ratio 2:3.



i) Show that the equation of the normal to the parabola at A is **3**

$$3x - 2ty = 2at^3 + 9at$$

ii) Deduce the coordinates of B **1**

iii) Determine the coordinates of C **2**

iv) Show that the locus of C is a parabola **2**

~ End of Paper ~

TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \quad = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

$NOTE: \quad \ln x = \log_e x, \qquad x > 0$

Yr 12 Extension One Trial HSC 2008 SOLUTIONS

Question One

a) $3^{x+1} = 2$

$\log 3^{x+1} = \log 2$

$(x+1)\log 3 = \log 2$.

$x+1 = \dfrac{\log 2}{\log 3}$

$x+1 = 0.63\ldots$

$x = -0.369\ldots$

$\therefore x = -0.37$ (2dp.)

b) $g(x) = \frac{1}{2}\cos^{-1}\frac{x}{2}$

$D: -1 \le \frac{x}{2} \le 1$

$\therefore -2 \le x \le 2$

$R: 0 \le y \le \frac{\pi}{2}$

c) $P(x) = 6x^3 + 17x^2 - 4x - 3$

$P(-3) = 6\times(-3)^3 + 17\times(-3)^2 - 4\times -3 - 3$

$= -162 + 153 + 12 - 3$

$= 0$

$\therefore x+3$ is a factor

$$\begin{array}{r} 6x^2 - x - 1 \\ x+3\,\overline{)\,6x^3 + 17x^2 - 4x - 3} \\ \underline{6x^3 + 18x^2} \\ -x^2 - 4x \\ \underline{-x^2 - 3x} \\ -x - 3 \\ \underline{-x - 3} \\ 0 \end{array}$$

$\therefore 6x^3 + 17x^2 - 4x - 3 = (x+3)(6x^2 - x - 1)$

$= (x+3)(3x+1)(2x-1)$

qu 1 d) $u = 2 - x^2$

$\frac{du}{dx} = -2x$

$\therefore x\,dx = \frac{-du}{2}$

$\int \frac{x\,dx}{(2-x^2)^3} = -\frac{1}{2}\int \frac{du}{u^3}$

$= -\frac{1}{2}\int u^{-3}\,du$

$= -\frac{1}{2}\,\frac{u^{-2}}{-2} + C$

$= \frac{1}{4(2-x^2)^2} + C$

e) $\frac{2x-5}{x-4} \ge x \qquad \therefore x \ne 4$

$(x-4)^2 \times \frac{(2x-5)}{x-4} \ge x(x-4)^2$

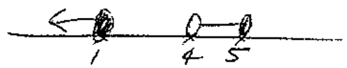
$(x-4)(2x-5) \ge x(x-4)^2$

$0 \ge x(x-4)^2 - (x-4)(2x-5)$

$0 \ge (x-4)[x(x-4) - 2x + 5]$

$0 \ge (x-4)(x^2 - 6x + 5)$

$0 \ge (x-4)(x-5)(x-1)$



if $x = 2 \qquad 0 \ge (-2)(-3)(1)$

$0 \ngeq 6$

qu 1 e) cont'd...

$\therefore \quad x \leq 1$ and $4 < x \leq 5$ (since $x \neq 4$)

Question Two

a) $\frac{d}{dx}\left(3x^2 \cos^{-1} x\right) = 6x \cos^{-1} x + 3x^2 \cdot \frac{-1}{\sqrt{1-x^2}}$

$= 6x \cos^{-1} x - \frac{3x^2}{\sqrt{1-x^2}}$

b) $\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{3} + \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{3} + \frac{\pi}{6}$

$= \frac{\pi}{2}$

c)

$y = 2x + 3 \qquad m_1 = 2$

$x - y = 1$

$y = x - 1 \qquad m_2 = 1$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2-1}{1+2\times 1}\right|$

$\tan\theta = \frac{1}{3}$

$\theta = 18°26'$

d) $(1+2x)^6(1-x)^4 = \left(\sum_{r=1}^{6}\binom{6}{r}1^r(2x)^{6-r}\right)\left(\sum_{r=1}^{4}\binom{4}{r}(-x)^{4-r}\right)$

$= \left(\sum_{r=1}^{6}\binom{6}{r}(2x)^{6-r}\right)\left(\sum_{r=1}^{4}\binom{4}{r}(-x)^{4-r}\right)$

$\binom{6}{3}(2x)^3 \times \binom{4}{4}(-x)^0 + \binom{6}{4}(2x)^2 \times \binom{4}{3}(-x)^1 +$

$\binom{6}{5}(2x)^1 \times \binom{4}{2}(-x)^2 + \binom{6}{6}(2x)^0\binom{4}{1}(-x)^3$

coefficient of x^3:

$8 \times \binom{6}{3} \times \binom{4}{4} - 4 \times \binom{6}{4}\binom{4}{3} + 2\binom{6}{5}\binom{4}{2} - \binom{6}{6}\binom{4}{1}$

$= -12.$

e) Test $n=1$,

$2^2+8 = 12$
$= 6\times 2$
$\therefore$ divisible by 6
$\therefore$ true for $n=1$

test $n=2$, $2^{2\times 2}+8 = 2^4+8$
$= 16+8$
$= 24$
$= 6\times 4$
$\therefore$ divisible by 6
$\therefore$ true for $n=2$

Assume true for $n=k$,

$2^{2k}+8 = 6M$ where M is a positive integer

Prove true for $n=k+1$,

$2^{2(k+1)}+8 = 6P$ where P is a positive integer

$LHS = 2^{2(k+1)}+8$
$= 2^{2k}.2^2+8$
$= (6M-8).4+8$ since $2^{2k}+8=6M$ $\therefore 2^{2k}=6M-8$
$= 24M-32+8$
$= 24M-24$
$= 6(4M-4)$
$= 6P$
$= RHS$

Since true for $n=1$ and $n=2$ and proved true for $n=k$ and $n=k+1$, statement is true for all $n\geq 1$.

Question Three

a) $\int_6^{15} \frac{dx}{\sqrt{x^2+64}} = \left[\ln\left(x+\sqrt{x^2+64}\right)\right]_6^{15}$

$= \ln\left(15+\sqrt{15^2+64}\right) - \ln\left(6+\sqrt{6^2+64}\right)$

$= \ln(15+17) - \ln(6+10)$

$= \ln 32 - \ln 16$

$= \ln \frac{32}{16}$

$= \ln 2$ as required.

b) i) $\angle XOZ = 32\times 2$ ($\angle$ at centre is twice $\angle$ at circumference on same arc)
$= 64°$

ii) $\angle XYZ = 180-32$ (opposite $\angle$'s of cyclic quad. WXYZ, supplementary)
$= 148°$

Qu 3 cont'd...

c) $4x^3 - 4x^2 - 29x + 15 = 0$.

Let the roots be α, β and $\alpha - \beta$

$\alpha + \beta + \alpha - \beta = \frac{-b}{a}$

$= \frac{+4}{4}$

$2\alpha = 1$

$\therefore \alpha = \frac{1}{2}$

$\alpha\beta(\alpha - \beta) = \frac{-d}{a}$

$\frac{1}{2}\beta\left(\frac{1}{2} - \beta\right) = \frac{-15}{4}$

$\frac{1}{4}\beta - \frac{1}{2}\beta^2 = \frac{-15}{4}$

$\beta - 2\beta^2 = -15$

$0 = 2\beta^2 - \beta - 15$

$0 = (2\beta + 5)(\beta - 3)$

$\beta = -\frac{5}{2}, 3$

$\therefore$ roots are $\frac{1}{2}, 3, -\frac{5}{2}$

Qu 3 cont'd...

d) i) LHS $= \frac{dN}{dt}$

$= \frac{d}{dt}\left(8000 + Ae^{kt}\right)$

$= Ake^{kt}$

RHS $= k(N - 8000)$

$= k\left(8000 + Ae^{kt} - 8000\right)$

$= Ake^{kt}$

$\therefore$ LHS = RHS.

ii) when $t = 0$, $N = 25000$

$25000 = 8000 + Ae^0$

$\therefore A = 25000 - 8000$

$= 17000$

when $t = 5$, $N = 29250$

$29250 = 8000 + 17000e^{5k}$

$21250 = 17000e^{5k}$

$1.25 = e^{5k}$

$5k = \ln 1.25$

$k = \frac{1}{5}\ln 1.25$

Question Four

a) i) $\frac{dV}{dt} = 12\ cm^3/s$

$\frac{15}{h} = \frac{15}{r}$

$\therefore h = r$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi r^3$

$\frac{dV}{dr} = \pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$12 = \pi r^2 \cdot \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{12}{\pi r^2}$

ii) $S = \pi r^2$

$\frac{dS}{dr} = 2\pi r$

$= 2\pi \times 5$

$= 10\pi$

$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$

$= 10\pi \times \frac{12}{\pi r^2}$

$= \frac{120}{r^2}$

When $r = 5$

$\frac{dS}{dt} = \frac{120}{5^2} = 4\cdot 8\ cm^2/s.$

Qu 4 cont'd...

b) i) by inspection at $x=0$, $\frac{dy}{dx} = 2$

ii) $y = 2\tan^{-1}x$

$\frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2}$

$\frac{1}{3} = \frac{2}{1+x^2}$

$6 = 1+x^2$

$x^2 = 5$

$\therefore x = \pm\sqrt{5}$

iii) $\int_{-\infty}^{\infty} \frac{2\,dx}{1+x^2} = 2\int_0^{\infty} \frac{2\,dx}{1+x^2}$

$= 4\int_0^{\infty} \frac{dx}{1+x^2}$

$= 4\left[\tan^{-1}x\right]_0^{\infty}$

$= 4 \times \pi/2$

$= 2\pi$

Question Five

a) $y = 2^x$

$\frac{dy}{dx} = \ln 2 . 2^x$

b) i) $\frac{d^2x}{dt^2} = 9(x-2)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 9(x-2)$

$\frac{1}{2}v^2 = 9\int x-2 \; dx$

$\frac{1}{2}v^2 = 9\left(\frac{x^2}{2} - 2x\right) + C$

initially, $x=4$ and $v=-6$

$\frac{1}{2}(36) = 9(8 - 2\times 4) + C$

$18 = 9\times 0 + C$

$\therefore C = 18$

$\frac{1}{2}v^2 = 9\left(\frac{x^2}{2} - 2x\right) + 18$

$v^2 = 9x^2 - 36x + 36$

$= 9(x^2 - 4x + 4)$

$\therefore v^2 = 9(x-2)^2$ as required

ii) from i) $v^2 = 9(x-2)^2$

$\therefore v = \pm 3(x-2)$

but $t=0$, $x=4$, $v=-6$

so $v = -3(x-2)$

$\frac{dx}{dt} = -3(x-2)$

$\frac{dt}{dx} = \frac{-1}{3(x-2)}$

$dt = \frac{1}{3}\int \frac{-dx}{x-2}$

$t = \frac{-1}{3}\ln(x-2) + C$

$t=0$, $x=4$

$0 = -\frac{1}{3}\ln(4-2) + C$

$0 = -\frac{1}{3}\ln 2 + C$

$\therefore C = \frac{1}{3}\ln 2$

$t = -\frac{1}{3}\ln(x-2) + \frac{1}{3}\ln 2$

$t = \frac{1}{3}\ln\left(\frac{2}{x-2}\right)$

$3t = \ln\frac{2}{x-2}$

$e^{3t} = \frac{2}{x-2}$

$x-2 = \frac{2}{e^{3t}}$

$x = 2 + 2e^{-3t}$

$\therefore x = 2(1+e^{-3t})$

Qu 5 cont'd...

b) iii)

from (ii) $v = -3(x-2)$

if $v = 0$, $0 = -3(x-2)$

$0 = x - 2$

$\therefore x = 2$

from (ii) also, $x = 2(1 + e^{-3t})$

if $x = 2$, $2 = 2(1 + e^{-3t})$

$1 = 1 + e^{-3t}$

$0 = e^{-3t}$

but $e^{-3t} \neq 0$ $\therefore v \neq 0$.

<u>alternate method</u> the graph of $x = 2(1+e^{-3t})$ has an asymptote at $x = 2$

so $x \neq 2$, $v \neq 0$.

qu5 cont'd...

c) i) $\angle RPS = \angle PQR$ (alternate segment theorem)

ie. The angle between tangent to circle and chord drawn from pt of contact is equal to angle in alternate segment

ii) $\angle RPS = \angle TPR$ (given PR bisects $\angle SPT$)

$\therefore \angle PQR = \angle TPR$ (from (i))

$\angle QPR = 90°$ ($\angle$ in semi-circle is right $\angle$)

$\therefore \angle QPT + \angle TPR = 90°$ (adjacent angles)

$\therefore \angle QPT + \angle PQT = 90°$ ($\angle PQT, \angle PQR$ same $\angle$)

$\therefore \angle PTQ = 90°$ ($\angle$ sum of ΔPQT)

$\therefore PT \perp QR$.

<u>or</u> $\angle RPS = \angle TPR$ (given PR bisects $\angle SPT$)

$= x$

$\therefore \angle PQR = \angle TPR$ (from (i))

$= x$

$\angle QPR = 90°$ ($\angle$ in semi-circle is right $\angle$'d)

$\therefore \angle TPQ = 90 - x$

$\angle TPQ + \angle PQT + \angle PTQ = 180°$ ($\angle$ sum of Δ)

$90 - x + x + \angle PTQ = 180°$

$\angle PTQ = 180 - x - (90 - x)$

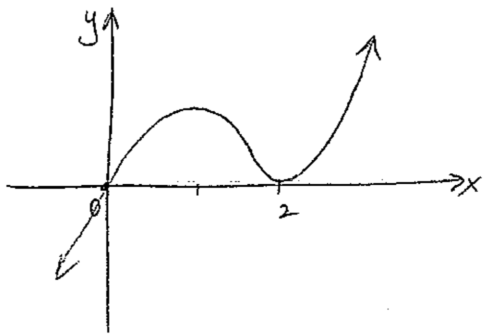
$= 180 - x - 90 + x$

$= 90°$

$\therefore PT \perp QR$

Question Six

a) i) $y = x(x-2)^2$



The inverse function $y = f^{-1}(x)$ exists if the graph $y = f(x)$ is one-to-one.

A horizontal line can be drawn to cut the graph at more than 1 pt.

One turning pt is $(2,0)$

The other is found by finding a stat. pt

$$y = x(x-2)^2$$
$$= x(x^2-4x+4)$$
$$= x^3 - 4x^2 + 4x$$
$$y' = 3x^2 - 8x + 4$$
$$= (3x-2)(x-2)$$
$$y' = 0$$

qu 6 b) i) cont'd...

$$0 = (3x-2)(x-2)$$
$$x = \tfrac{2}{3}, 2.$$

$\therefore y = f^{-1}(x)$ exists if $a \le \frac{2}{3}$.

ii) D. for $y = f(x)$ $x \le \frac{2}{3}$

R: $f(\frac{2}{3}) = \frac{2}{3}(\frac{2}{3}-2)^2$
$$= \frac{32}{27}$$

$\therefore$ D: for $y = f^{-1}(x)$ is $x \le \frac{32}{27}$.

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b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 3x \, dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 - \cos 6x \, dx$

$$\cos 2x = 1 - 2\sin^2 x$$
$$\cos 6x = 1 - 2\sin^2 3x$$
$$2\sin^2 3x = 1 - \cos 6x$$
$$\sin^2 3x = \tfrac{1}{2}(1 - \cos 6x)$$

$$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \frac{1}{2}\left\{\frac{\pi}{3} - \frac{1}{6}\sin 2\pi - \left(\frac{\pi}{4} - \frac{1}{6}\sin\frac{3\pi}{2}\right)\right\}$$
$$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{4} - \frac{1}{6}\right]$$
$$= \frac{1}{2}\left[\frac{4\pi - 3\pi}{12} - \frac{1}{6}\right]$$
$$= \frac{1}{2}\left[\frac{\pi}{12} - \frac{1}{6}\right] \text{ as required.}$$

qu 6 cont'd...

c) i) $y = vt\sin 40° - 5t^2$

ii) $x = vt\cos 40°$

iii) from ii) $t = \frac{x}{v\cos 40°}$

$y = v \cdot \frac{x}{v\cos 40°} \cdot \sin 40° - 5\left(\frac{x}{v\cos 40°}\right)^2$

$= x\tan 40° - \frac{5x^2}{v^2\cos^2 40°}$

$= x\tan 40° - \frac{5x^2}{v^2}(\sec^2 40°)$

$\therefore y = x\tan 40° - \frac{5x^2}{v^2}(1 + \tan^2 40°)$

iv) $x = 20\text{ m}, y = 6\text{ m}$

$6 = 20\tan 40° - \frac{5(20^2)}{v^2}(1 + \tan^2 40°)$

$6 = 20\tan 40° - \frac{2000}{v^2}(1 + \tan^2 40°)$

$\frac{2000}{v^2}(1 + \tan^2 40°) = 20\tan 40° - 6$

$\frac{2000(1 + \tan^2 40°)}{20\tan 40° - 6} \quad v^2$

$\therefore v = 17.8\text{ m/s}$

Question Seven

a) $\int_0^{1.25} \frac{5\,dx}{\sqrt{25 - 16x^2}} = \left[\frac{5}{4}\sin^{-1}\frac{4x}{5}\right]_0^{1.25}$

$= \frac{5}{4}\left(\sin^{-1}\frac{4}{5}\times 1.25 - \sin^{-1}0\right) = \frac{5}{4}\times\frac{\pi}{2} = \frac{5\pi}{8}$

b) $e^x - x - 2 = 0$

$f(x) = e^x - x - 2 \qquad f(1.2) = e^{1.2} - 3.2$

$f'(x) = e^x - 1 \qquad f'(1.2) = e^{1.2} - 1$

$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$ where $a_1 = 1.2$

$= 1.2 - \frac{(e^{1.2} - 3.2)}{(e^{1.2} - 1)}$

$= 1.148\ldots$

$a_2 \doteqdot 1.15$ (2 d.p.)

qu 7 continued...

c) i) $x^2 = -9ay$

$y = -\frac{x^2}{9a}$

$y' = \frac{-2x}{9a}$

at A, $x = 3at$

$m_T = \frac{-6at}{9a} = \frac{-2t}{3}$

$\therefore m_N = \frac{3}{2t}$

$y - y_1 = m(x - x_1)$

$y + at^2 = \frac{3}{2t}(x - 3at)$

$2ty + 2at^3 = 3x - 9at$

N: $3x - 2ty = 2at^3 + 9at$

ii) at B, $x = -at$

subst. $x = -at$ into eq'n in (i)

$3(-at) - 2ty = 2at^3 + 9at$

$-2ty = 2at^3 + 12at$

$y = -at^2 - 6a$

$\therefore B(-at, -at^2 - 6a)$

qu 7 c) continued...

iii) $(3at, -at^2)$ $(-at, -at^2 - 6a)$ $-2:3$

$x = \frac{3at \times 3 + -at \times -2}{-2+3}$

$= \frac{9at + 2at}{1}$

$\therefore x = 11at$

$y = \frac{3 \times -at^2 + -2(-at^2 - 6a)}{-2+3}$

$= -3at^2 + 2at^2 + 12a$

$y = 12a - at^2$

$\therefore C(11at, 12a - at^2)$

iv) from iii)

$x = 11at$

$t = \frac{x}{11a}$

$y = 12a - a\left(\frac{x}{11a}\right)^2$

$= 12a - \frac{a \cdot x^2}{121a^2}$

$y - 12a = \frac{-x^2}{121a}$

$121a(y - 12a) = -x^2$

$\therefore x^2 = -121a(y - 12a)$

$\therefore$ the locus of C is a parabola.